Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 258 (2010) 2884–2909

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Restriction estimates for some surfaces with vanishing curvatures ☆

Sanghyuk Lee [a,*], Ana Vargas [b]

[a] *School of Mathematical Sciences, Seoul National University, Seoul 151-742, Republic of Korea*
[b] *Department of Mathematics, University Autonoma de Madrid, 28049 Madrid, Spain*

Received 18 June 2009; accepted 13 January 2010

Available online 11 February 2010

Communicated by J. Bourgain

Abstract

We obtain bilinear restriction estimates for surfaces with vanishing curvatures. As application we also prove new linear restriction estimates for some class of conic surfaces.

Keywords: Restriction estimates; Conic surfaces

1. Introduction

In this note we consider Fourier restriction estimates for some class of conic surfaces. It has been known that the curvature plays an important role in determining the boundedness of the restriction operators. Let S be a smooth compact surface in $\mathbb{R}^{n+1}$ with the induced Lebesgue measure $d\sigma$. It is well known [3,7,11] that if k principal curvatures are nonzero at each point of the surface S, for $q \geqslant \frac{2k+4}{k}$

$$\|\widehat{f\,d\sigma}\|_{L^q(\mathbb{R}^{n+1})} \leqslant C\|f\|_{L^2(d\sigma)}. \tag{1.1}$$

☆ The first author was partially supported by the NRF of Korea grant (2009-0072531). The second author was partially supported by the grant MTM2007-60952 (Spain).

* Corresponding author.
E-mail addresses: shklee@snu.ac.kr (S. Lee), ana.vargas@uam.es (A. Vargas).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.01.014

The range on q is optimal as it can be easily seen using Knapp's example. The natural conjecture is that the estimate

$$\|\widehat{f\,d\sigma}\|_{L^q(\mathbb{R}^{n+1})} \leqslant C\|f\|_{L^p(d\sigma)}$$

holds if $\frac{k+2}{q} \leqslant k(1-\frac{1}{p})$ and $p > \frac{2k+2}{k}$. It was Bourgain [1] who first obtained a result beyond the sharp L^2–L^q restriction estimates when $k \geqslant 2$.

Recent development on the restriction problem has been made by considering a suitable bilinear version of the operator [5,8–10,12,13]. To be specific, let S_1, S_2 be subsets of S with measures $d\sigma_1$, $d\sigma_2$. Let us consider the bilinear adjoint restriction estimate

$$\|\widehat{f\,d\sigma_1}\widehat{g\,d\sigma_2}\|_{L^p} \leqslant C\|f\|_{L^2(d\sigma_1)}\|g\|_{L^2(d\sigma_2)}. \tag{1.2}$$

In this form of estimate one can impose additional conditions which specify the relative position of the two surfaces. One typical condition is *transversality*. For positively curved surfaces (e.g. the cone and the paraboloid) transversality makes it possible to get a wider range of boundedness than is allowed for the linear estimates. However, for the surfaces with positive and negative principal curvatures transversality is not enough to obtain such improvement. Actually one needs stronger (separation) conditions [5,12]. Unlike the case of linear estimate (1.1), the role of curvature in bilinear restriction estimates does not seem to be clearly understood. In fact, the sharp bilinear restriction estimates (1.2) for the cone and the paraboloid are valid for the same range of q except the endpoint even though the cone has only $n-1$ nonzero principal curvatures.

The estimate (1.2) has been studied mainly with surfaces with nonvanishing Gaussian curvature or one vanishing curvature. In this note we want to generalize the known bilinear restriction estimates to the surfaces having two or more vanishing curvatures.

Let $k \leqslant n-1$ be an integer. We assume that S has k nonvanishing principal curvatures and $n-k$ vanishing curvatures. In other words the surface S has $n-k$ null directions along which the curvatures vanish. To be more precisely, let $\pm\mathbf{N}(\xi) \in \mathbb{S}^n$ be the unite normal vector of S at ξ. By rotation and decomposition we may assume that $|\mathbf{N}(\xi) - e_{n+1}| \leqslant 1/2$.

Definition 1.1. Suppose that S is a smooth compact surface with (possibly) boundary in $\mathbb{R}^{n+1}$. We say that S is of conic type with k nonvanishing curvatures if the following assumptions are satisfied:

- The map $d\mathbf{N} : T_\xi(S) \to T_{N(\xi)}(\mathbb{S}^n)$ has k nonzero eigenvalues and $n-k$ zero eigenvalues.
- The nonzero eigenvalues have magnitude ~ 1.[1]

We denote by $\mathcal{N}_\xi(S)$ the span of the eigenvectors with zero eigenvalues of $d\mathbf{N}$ at ξ. We say that any nonzero vector v is in a null direction of S at ξ if $v \in \mathcal{N}_\xi(S)$.

Definition 1.2. Let S_1, S_2 be subsets of a conic type surface S of k nonvanishing curvatures. We say that S_1 and S_2 are transversal if

[1] For $A, B > 0$, $A \sim B$ means $C^{-1}A \leqslant B \leqslant CA$ for some constant $C > 0$.

$$\left|\mathbf{N}(\xi_1) - \mathbf{N}(\xi_2)\right| \sim 1 \tag{1.3}$$

for $\xi_1 \in S_1$ and $\xi_2 \in S_2$.

Let z_1, z_2 be points in $\mathbb{R}^{n+1}$ and let us denote the translated surfaces by

$$S_{z_j} = S_j + z_j,^{2} \quad j = 1, 2.$$

Then by the condition (1.3) we may assume that the intersection of two surfaces S_{z_1} and S_{z_2} is a smooth $(n-1)$-dimensional immersed submanifold by dividing the surfaces into small pieces (if necessary), as long as the intersection is not empty. We denote the intersection by

$$\Pi_{z_1,z_2} = S_{z_1} \cap S_{z_2}.$$

As it is well known, the dispersion of the normal vectors of a given surface accounts for the decay of Fourier transform of the surface measure, which was crucial in obtaining the linear restriction estimate (1.1). As it turns out, for the bilinear estimates the dispersion along the intersection Π_{z_1,z_2} is important. Roughly, the number of nonzero curvatures along Π_{z_1,z_2} takes the role that is played by the total number of nonzero curvatures in the linear estimates. (See Theorem 1.4 below.) This explains why the range of bilinear restriction estimates for the cone and paraboloid is essentially the same.

Let us denote by $T_\xi(\Pi_{z_1,z_2})$ the tangent space of Π_{z_1,z_2} at ξ. Now we make an assumption on the surfaces S_1 and S_2:

$$\begin{aligned}\dim\bigl(T_\xi(\Pi_{z_1,z_2}) \oplus \mathcal{N}_{\xi_1}(S_1)\bigr) = n,\\ \dim\bigl(T_\xi(\Pi_{z_1,z_2}) \oplus \mathcal{N}_{\xi_2}(S_2)\bigr) = n\end{aligned} \tag{1.4}$$

for all $\xi \in \Pi_{z_1,z_2}$, $\xi_1 \in S_1$ and $\xi_2 \in S_2$ as long as $\Pi_{z_1,z_2} \neq \emptyset$. This is one of most important assumption which gives the maximal amount of dispersion of normal vectors along the intersection Π_{z_1,z_2}. Since the surface S has k nonvanishing principal curvatures, the maps $\mathbf{N}_{z_i} : S_{z_i} \to \mathbb{S}^n$ have rank k. Here $\mathbf{N}_{z_j}(\xi)$ is the normal vector to S_{z_i} at ξ. Hence from the condition (1.4) one can easily see that for $j = 1, 2$, the map $\mathbf{N}^j_{z_1,z_2}$ which is given by

$$\xi \in \Pi_{z_1,z_2} :\to \mathbf{N}_{z_j}(\xi) \in \mathbb{S}^n$$

is also of rank k. That is to say, the rank of $d\mathbf{N}^j_{z_1,z_2}$ is k, $j = 1, 2$.

Finally we assume that

$$\begin{aligned}\mathbf{N}_{z_2}(\xi_2) \notin d\mathbf{N}^1_{z_1,z_2}\bigl(T_\xi(\Pi_{z_1,z_2})\bigr) \oplus \operatorname{span}\bigl\{\mathbf{N}_{z_1}(\xi_1)\bigr\},\\ \mathbf{N}_{z_1}(\xi_1) \notin d\mathbf{N}^2_{z_1,z_2}\bigl(T_\xi(\Pi_{z_1,z_2})\bigr) \oplus \operatorname{span}\bigl\{\mathbf{N}_{z_2}(\xi_2)\bigr\}\end{aligned} \tag{1.5}$$

[2] Obviously, multiplication of $e^{-2\pi i x\xi_0}$ to $\widehat{f\,d\sigma_j}$ does not have any effect on the estimate and the Fourier transform of $e^{-2\pi i x\xi_0}\widehat{f\,d\sigma_j}$ is supported in $S_j + \xi_0$. So, it is natural to consider conditions which are valid uniformly for the translated surfaces.

for $\xi \in \Pi_{z_1,z_2}$, $\xi_1 \in S_{z_1}$ and $\xi_2 \in S_{z_2}$ as long as $\Pi_{z_1,z_2} \neq \emptyset$. For the positively curved surfaces (e.g. the cone, sphere, or paraboloid) this type of transversality can be obtained by the normal separation condition (1.3) but it is not the case for the surfaces with principal curvatures of different sings. This is actually the separation condition which was used to obtain the best possible bilinear restriction estimates for the hyperboloid [5,12].

Remark 1.3. The condition (1.5) is concerned with the transversality between the cone generated by the normal vectors from the intersection surface Π_{z_1,z_2} and the normal vectors to the opposite surface. More precisely, for $j = 1, 2$, let us set

$$\Gamma_j = \big\{ t\mathbf{N}_{z_j}(\xi)\colon\ \xi \in \Pi_{z_1,z_2},\ 1 \leqslant |t| \leqslant 2 \big\}.$$

Then the condition (1.5) equivalently means that any normal vector $\mathbf{N}_1$ ($\mathbf{N}_2$, resp.) of S_{z_1} (S_{z_2}, resp.) is transversal to Γ_2 (Γ_1, resp.) because the tangent space of Γ_j is given by $d\mathbf{N}^j_{z_1,z_2}(T_\xi(\Pi_{z_1,z_2})) \oplus \operatorname{span}\{\mathbf{N}_{z_j}(\xi)\}$.

The following is our main result:

Theorem 1.4. *Let $1 \leqslant k \leqslant n-1$. Suppose that S is a smooth compact surface of conic type in $\mathbb{R}^{n+1}$ with k-nonvanishing curvatures. If the surfaces $S_1, S_2 \subset S$ satisfy the assumptions* (1.3), (1.4) *and* (1.5)*, then for $p > \frac{k+4}{k+2}$*

$$\|\widehat{f\,d\sigma_1}\widehat{g\,d\sigma_2}\|_{L^p} \leqslant C\|f\|_2\|g\|_2.$$

This theorem is sharp in the sense that there are surfaces satisfying (1.3), (1.4) and (1.5) but the estimate fails for $p < \frac{k+4}{k+2}$. See Remark 3.3 and the proof of Proposition 3.2.

Remark 1.5. If one considers two transversal subsets of a cylinder in $\mathbb{R}^3$ satisfying (1.3), then the condition (1.5) is trivially satisfied. But Π_{z_1,z_2} is contained in a line which is parallel with the null direction. Hence (1.4) fails. As it can be easily seen, the best possible bilinear restriction L^2 estimate is $L^2 \times L^2 \to L^2$ estimate. Also, considering the n-dimensional cylinder $(\xi', \xi'', \sqrt{1-|\xi'|^2})$, $(\xi', \xi'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ and suitable transversal subsets S_1 and S_2 it is easy to see that (1.2) fails for $p < \frac{k+3}{k+1}$ even though the conditions (1.3) and (1.5) are satisfied. On the other hand, if we only assume the conditions (1.3) and (1.5) without (1.4), then one can show for $p > \frac{k+3}{k+1}$[3]

$$\|\widehat{f\,d\sigma_1}\widehat{g\,d\sigma_2}\|_{L^p} \leqslant C\|f\|_2\|g\|_2.$$

But this is still better than the trivial $L^2 \times L^2 \to L^{\frac{k+2}{k}}$ estimate which follows from the linear estimate (1.1).

[3] This can be shown by following the argument below. The only thing one has to observe is that the combinatorial estimates in Lemma 2.3 lose an additional factor of $R^{\frac{1}{2}}$ without (1.4).

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.4. As application we consider some model surfaces of conic type in Section 3 and obtain new restriction estimates.

2. Proof of Theorem 1.4

This section is devoted to proving Theorem 1.4. The proof here is based on the induction on scale argument which was used to obtain the sharp bilinear restriction estimates [8,13] (also see [5,12]). However adaptations are needed to reflect the geometry of conic surfaces.

By rotation, translation and breaking the surfaces S into surfaces of small diameter if necessary, we may assume that the surface S is given by the graph of a function. For a $\delta_0 > 0$ let ϕ be a smooth function such that $\phi : Q = [-\delta_0, \delta_0]^n \to \mathbb{R}$ and ϕ satisfies

$$\phi(0) = 0, \qquad \nabla\phi(0) = 0.$$

We may also assume that the surfaces S_1, S_2 are given by the graphs of the function $-\phi$ over the set $Q_1, Q_2 \subset Q$, respectively. That is to say, for $j = 1, 2$,

$$S_j = \big\{\big(x, -\phi(x)\big)\colon\ x \in Q_j\big\}$$

for some cubes $Q_1, Q_2 \subset Q$. Then for $j = 1, 2$, let us define the extension operators

$$\mathcal{E}_j f(x,t) = \int\limits_{Q_j} e^{i(x\cdot\xi - t\phi(\xi))} f(\xi)\, d\xi. \tag{2.1}$$

Since $d\sigma_i$ is comparable to $d\xi$, it is enough to show that for $p > \frac{k+4}{k+2}$,

$$\Bigg\| \prod_{j=1}^{2} \mathcal{E}_j f_j \Bigg\|_p \leqslant C \prod_{j=1}^{2} \|f_j\|_2.$$

First, we decompose $\mathcal{E}_j f$ into a sum of *wave packets*. The wave packets have good localization properties in both Fourier transform side and (x,t)-space. Unlike the usual decomposition of an $O(R^{-1})$-neighborhood of a conic surface [2,6,13], which takes into account the null directions, we use a more direct decomposition in spirit of [8] (also see [5]). It does not depend on the presence of null directions.

2.1. Wave packet decomposition at scale R

For $d > 0$ and $A \subset \mathbb{R}^n$, let us denote by $A + O(d)$ the set

$$\big\{x \in \mathbb{R}^n\colon\ \mathrm{dist}(x, A) < Cd\big\},$$

for some big constant $C > 0$.

Let $R \gg 1$. The wave packet decomposition at scale R makes the support of the functions be expanded by $O(R^{-1/2})$ (see Lemma 2.1). So, we need to consider a little bit larger sets than Q_1, Q_2. For $CR^{-1/2} < \epsilon \ll \delta_0$, let us set

$$V_j = Q_j + O(\epsilon).$$

We define the spatial grid $\mathcal{Y}$ by

$$\mathcal{Y} = R^{1/2}\mathbb{Z}^n$$

and the frequency grids $\mathcal{V}_1$, $\mathcal{V}_2$, respectively by setting

$$\mathcal{V}_j = R^{-1/2}\mathbb{Z}^n \cap V_j, \quad j = 1, 2.$$

Let us set

$$W_j = \bigl\{(y, v):\ (y, v) \in \mathcal{Y} \times \mathcal{V}_j\bigr\}.$$

For $w_j = (y_j, v_j) \in W_j$, we define the associated tube T_{w_j} by

$$T_{w_j} = \bigl\{(x, t) \in \mathbb{R}^n \times \mathbb{R}:\ |t| \leqslant 2R,\ \bigl|x - \bigl(y_j + t\nabla\phi(v_j)\bigr)\bigr| \leqslant R^{1/2}\bigr\}.$$

Obviously T_{y_j,v_j} contains $(y_j, 0)$ and its major direction is parallel to $(\nabla\phi(v_j), 1) \in \mathbb{R}^n \times \mathbb{R}$, which is parallel to the normal vector of the surface S_j at $(v_j, \phi(v_j))$. That is,

$$\bigl(\nabla\phi(v_j), 1\bigr) \parallel \mathbf{N}_{(v_j,\phi(v_j))}. \tag{2.2}$$

The following is a modification of Lemma 4.1 in [8]. For a proof see [5].

Lemma 2.1 *(Wave packet decomposition). Let ϕ be a smooth function defined on Q. Suppose that f_1, f_2 are supported in Q_1, Q_2, respectively. If $|t| \leqslant 10R$, we can write $\mathcal{E}_j f_j$ as*

$$\mathcal{E}_j f_j(x, t) = \sum_{w_j \in W_j} C_{w_j} p_{w_j}(x, t), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

such that C_{w_j}, p_{w_j} satisfy the following conditions:

(P1) *For $j = 1, 2$,*

$$\Bigl(\sum_{w_j \in W_j} |C_{w_j}|^2\Bigr)^{1/2} \leqslant C\|f_j\|_2.$$

(P2) *For $j = 1, 2$,*

$$p_{w_j} = \mathcal{E}_j\bigl(\widehat{p_{w_j}(\cdot, 0)}\bigr).$$

(P3) *If* $w_j = (y_j, v_j)$, *then*

$$\operatorname{supp} \widehat{p_{w_j}(\cdot, t)} \subset \{\xi \colon \xi = v_j + O(R^{-1/2})\}.$$

(P4) *For any* $N > 0$, $|t| \leqslant 10R$,

$$|p_{y_j,v_j}(x,t)| \leqslant C_N R^{-n/4}\left(1 + \frac{|x - (y_j + t\nabla\phi(v_j))|}{R^{1/2}}\right)^{-N}.$$

In particular, if $\operatorname{dist}((x,t), T_{y_j,v_j}) \geqslant R^{\delta+1/2}$, *then*

$$|p_{y_j,v_j}(x,t)| \leqslant C R^{-100n}.$$

(P5) *For any* $S \subset W_j$,

$$\left\| \sum_{w_j \in S} p_{w_j}(\cdot, t) \right\|_2^2 \leqslant C\#S.$$

2.2. Induction on scale

Since $k \geqslant 1$, the Fourier transform of the surface measures have some decay at infinity. Hence, by the globalization Lemma 2.4 in [9], it is enough to show that for any $\alpha > 0$

$$\left\| \prod_{i=1}^{2} \mathcal{E}_j f_j \right\|_{L^{\frac{k+4}{k+2}}(Q(R))} \leqslant C R^{\alpha} \prod_{i=1}^{2} \|f_j\|_2. \tag{2.3}$$

Here $Q(R)$ is the cube which is centered at the origin and of side length R. Let us denote the estimates (2.3) by $\mathcal{E}^*(\alpha)$.

We may assume $\|f_1\|_2 = \|f_2\|_2 = 1$. Since $|\mathcal{E}_i f_i(x,t)| \leqslant C\|f\|_2$, using the wave packet decomposition and the standard pigeonholing argument together with the property (P1) in Lemma 2.1 the proof of (2.3) reduces (modulo loss of $(\log R)^2$ in bounds[4]) to obtaining the estimate

$$\left\| \sum_{w_1 \in \mathcal{W}_1,\, w_2 \in \mathcal{W}_2} p_{w_1} p_{w_2} \right\|_{L^{\frac{k+4}{k+2}}(Q(R))} \leqslant C R^{\alpha} |\mathcal{W}_1|^{1/2} |\mathcal{W}_2|^{1/2} \tag{2.4}$$

for any subsets $\mathcal{W}_1 \subset W_1$ and $\mathcal{W}_2 \subset W_2$ whenever p_{w_j} is the L^2 normalized wave packet satisfying (P2), (P3), (P4) and (P5). By rapid decay of p_{w_j} away from the associated tube T_{w_j} we may assume that T_{w_j} meets with $Q(2R)$. The main part of the induction on scale argument is to establish the following implication:

[4] Such loss is harmless due to the nature of the estimate.

If (2.4) is valid for $R \gg 1$, then for $0 < \delta \ll 1$,

$$\Big\| \sum_{w_1 \in \mathcal{W}_1,\, w_2 \in \mathcal{W}_2} p_{w_1} p_{w_2} \Big\|_{L^{\frac{k+4}{k+2}}(Q(R))} \leqslant C R^{\alpha(1-\delta)+c\delta} |\mathcal{W}_1|^{1/2} |\mathcal{W}_2|^{1/2} \tag{2.5}$$

with c independent of R and δ. Hence we have the implication

$$\mathcal{E}^*(\alpha) \to \mathcal{E}^*\big(\alpha(1-\delta)+c\delta\big)^5$$

for any $\alpha > 0$. Choosing sufficiently small δ and iterating this estimates finitely many times one can show that $R^*(\alpha)$ is valid for any $\alpha > 0$.

Pigeonholing further we can specify some of the quantities involved. Let us partition $Q(2R)$ into disjoint $R^{1/2}$-cubes q and denote the collection of those cubes by $\mathcal{Q}(R)$. We classify the q, w_1 and w_2 using dyadic parameters. For each $q \in \mathcal{Q}$ let us set

$$\mathcal{W}_j(q) = \big\{ w_j \in \mathcal{W}_j\colon\ R^\delta q \cap T_{w_j} \neq \emptyset \big\}.$$

Here $R^\delta q$ is the set having the same center as q and expanded to R^δ times from q. We also set for each dyadic number $\mu_1, \mu_2 \geqslant 1$,

$$\mathcal{Q}^{\mu_1,\mu_2} = \big\{ q\colon\ \big|\mathcal{W}_j(q)\big| \sim \mu_j,\ j = 1, 2 \big\}.$$

For each w_j, define

$$\mathcal{Q}^{\mu_1,\mu_2}(w_j) = \big\{ q \in \mathcal{Q}^{\mu_1,\mu_2}\colon\ R^\delta q \cap w_j \neq \emptyset \big\}$$

and for dyadic $\lambda \geqslant 1$, we again set

$$\mathcal{W}_j^{\lambda,\mu_1,\mu_2} = \big\{ w_j \in \mathcal{W}_j\colon\ \big|\mathcal{Q}^{\mu_1,\mu_2}(w_j)\big| \sim \lambda \big\}.$$

Then by (P4) in Lemma 2.1 and pigeonholing, it is easy to see

$$\begin{aligned}
&\Big\| \sum_{w_1 \in \mathcal{W}_1,\, w_2 \in \mathcal{W}_2} p_{w_1} p_{w_2} \Big\|_{L^{\frac{k+4}{k+2}}(Q(R))} \\
&\quad \leqslant C(\log R)^4 \Big\| \sum_{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2},\, j=1,2} \Big(\sum_{q \in \mathcal{Q}^{\mu_1,\mu_2}} \chi_q p_{w_1} p_{w_2} \Big) \Big\|_{L^{\frac{k+4}{k+2}}(Q(R))} + O\big(R^{-M}\big)
\end{aligned}$$

for some dyadic $1 \leqslant \lambda_1, \lambda_2, \mu_1, \mu_2 \leqslant R^C$ and large M. Hence the matter is reduced to showing

5 The losses of $(\log R)^C$ are absorbed in $R^{c\delta}$.

$$\Big\| \sum_{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2},\, j=1,2} \Big(\sum_{q \in Q^{\mu_1,\mu_2}} \chi_q p_{w_1} p_{w_2} \Big) \Big\|_{L^{\frac{k+4}{k+2}}(Q(R))}$$
$$\leqslant C\big(R^{\alpha(1-\delta)+c\delta}\big)|\mathcal{W}_1|^{1/2}|\mathcal{W}_2|^{1/2}. \tag{2.6}$$

Now we fix the numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$ for the rest of this section. We partition $Q(2R)$ into $O(R^{(n+1)\delta})$ cubes b which are of side length $R^{1-\delta}$ and essentially disjoint. So, we get

$$\Big\| \sum_{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2},\, j=1,2} \Big(\sum_{q \in Q^{\mu_1,\mu_2}} \chi_q p_{w_1} p_{w_2} \Big) \Big\|_{L^{\frac{k+4}{k+2}}(Q(R))}$$
$$\leqslant \sum_b \Big\| \sum_{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2},\, j=1,2} \Big(\sum_{q \in Q^{\mu_1,\mu_2},\, q \subset 2b} \chi_q p_{w_1} p_{w_2} \Big) \Big\|_{L^{\frac{k+4}{k+2}}(b)}.$$

For each w_j, let $b(w_j)$ be the cube b which contains the maximal number of $q \in \mathcal{Q}^{\mu_1,\mu_2}(w_j)$. There may be many of such cubes but we simply choose one of them. We now define a relation $\backsim$ between b and $w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}$ by saying

$$b \backsim w_j \quad \text{if } b \cap 10b(w_j) \neq \emptyset.$$

Since the number of b is $O(R^{(n+1)\delta})$, it is obvious that

$$\big|\big\{q \in \mathcal{Q}^{\mu_1,\mu_2}\colon\ w_j \cap R^\delta q \neq \emptyset,\ q \cap 10b = \emptyset\big\}\big| \gtrsim R^{-c\delta}\lambda_j$$

provided $w_j \nsim b$. Then

$$\sum_b \Big\| \sum_{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2},\, j=1,2} \Big(\sum_{q \in Q^{\mu_1,\mu_2},\, q \subset 2b} \chi_q p_{w_1} p_{w_2} \Big) \Big\|_{L^{\frac{k+4}{k+2}}(b)}$$
$$\leqslant \sum_b \big(I^{\backsim b} + I^{\nsim b}\big),$$

where

$$I^{\backsim b} = \Big\| \sum_{w_1 \backsim b,\, w_2 \backsim b} \Big(\sum_{q \in Q^{\mu_1,\mu_2},\, q \subset 2b} \chi_q p_{w_1} p_{w_2} \Big) \Big\|_{L^{\frac{k+4}{k+2}}(b)},$$
$$I^{\nsim b} = \Big\| \sum_{w_1 \nsim b,\ \text{or}\ w_2 \nsim b} \Big(\sum_{q \in Q^{\mu_1,\mu_2},\, q \subset 2b} \chi_q p_{w_1} p_{w_2} \Big) \Big\|_{L^{\frac{k+4}{k+2}}(b)}.$$

To simplify the notation, in the summations $\sum_{w_1 \backsim b,\, w_2 \backsim b}$, $\sum_{w_1 \not\backsim b, \text{ or } w_2 \not\backsim b}$, (w_1, w_2) is assumed to be in the set $\mathcal{W}_1^{\lambda_1,\mu_1,\mu_2} \times \mathcal{W}_2^{\lambda_2,\mu_1,\mu_2}$. Therefore we are reduced to showing

$$\sum_b \left(I^{\backsim b} + I^{\not\backsim b}\right) \leqslant C\left(R^{\alpha(1-\delta)+c\delta}\right)|\mathcal{W}_1|^{1/2}|\mathcal{W}_2|^{1/2}.$$

From the induction hypothesis (2.3) and (P5), it follows that

$$I^{\backsim b} \leqslant C R^{\alpha(1-\delta)} \prod_{i=1}^{2} \left|\left\{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}\colon\, b \backsim w_j\right\}\right|^{1/2}$$

since the side length of b is $\sim R^{1-\delta}$. By Cauchy–Schwarz's inequality it follows that

$$\begin{aligned}\sum_b I^{\backsim b} &\leqslant C R^{\alpha(1-\delta)} \prod_{i=1}^{2} \left(\sum_b \left|\left\{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}\colon\, b \backsim w_j\right\}\right|\right)^{1/2} \\ &\leqslant C R^{\alpha(1-\delta)} \prod_{i=1}^{2} \left(\sum_{w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}} \left|\{b\colon\, b \backsim w_j\}\right|\right)^{1/2} \\ &\leqslant C R^{\alpha(1-\delta)} \prod_{i=1}^{2} |\mathcal{W}_j|^{1/2}.\end{aligned}$$

For the second inequality we use the fact that there are only $O(1)$ cubes b with $b \backsim w_j$ for $w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}$. The induction assumption is used to handle highly concentrating part and under proper relation this gives slightly improved bound because the considered cube is of size $R^{1-\delta}$. It is in fact the major advantage of the induction scale argument. However, we have to trade off such easiness of improvement with a detailed analysis when we handle less concentrating part.

Now we need to show

$$I^{\not\backsim b} \leqslant C R^{c\delta}|\mathcal{W}_1|^{1/2}|\mathcal{W}_2|^{1/2}.$$

It follows from interpolation between the easy L^1-estimate, $I^{\not\backsim b} \leqslant CR|\mathcal{W}_1|^{1/2}|\mathcal{W}_2|^{1/2}$[6] and the L^2-estimate

$$\left\| \sum_{w_1 \not\backsim b, \text{ or } w_2 \not\backsim b} \left(\sum_{q \in Q^{\mu_1,\mu_2},\, q \subset 2b} \chi_q p_{w_1} p_{w_2} \right) \right\|_{L^2(b)}^2 \leqslant C R^{-k/2+c\delta}|\mathcal{W}_1||\mathcal{W}_2|. \tag{2.7}$$

Here the summation is taken over $w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}$, $j = 1, 2$. Hence it remains to show (2.7).

[6] This follows from the trace lemma, or (P5).

2.3. L^2 estimates over $R^{1/2}$-cubes

For given $v_1 \in \mathcal{V}_1$ and $v_2' \in \mathcal{V}_2$, let us set

$$z_1 = \big(v_2', \phi\big(v_2'\big)\big), \qquad z_2 = \big(v_1, \phi(v_1)\big)$$

and also set

$$\widetilde{\Pi}_{z_1,z_2} = \bigcap_{i=1}^{2} \big(S_j + z_j + O\big(R^{-1/2}\big)\big)$$

which also can be considered as $\Pi_{z_1,z_2} + O(R^{-1/2})$. For $\mathcal{U}_j \subset W_j$, $j = 1, 2$, let us set

$$\mathcal{U}_j^{\widetilde{\Pi}_{z_1,z_2}} = \big\{w_j = (y_j, v_j) \in \mathcal{U}_j \colon \big(v_j, \phi(v_j)\big) + z_j \in \widetilde{\Pi}_{z_1,z_2}\big\}.$$

Lemma 2.2. *Let q be an $R^{1/2}$-cube and $\mathcal{U}_j \subset \mathcal{W}_j(q)$, $j = 1, 2$. Then,*

$$\bigg\| \sum_{w_1 \in \mathcal{U}_1} \sum_{w_2 \in \mathcal{U}_2} p_{w_1} p_{w_2} \bigg\|_{L^2}^2 \lesssim R^{c\delta} R^{-(n-1)/2} |\mathcal{U}_1||\mathcal{U}_2| \min\Big(\sup_{z_1,z_2} \big|\mathcal{U}_1^{\widetilde{\Pi}_{z_1,z_2}}\big|, \sup_{z_1,z_2} \big|\mathcal{U}_2^{\widetilde{\Pi}_{z_1,z_2}}\big|\Big).$$

Proof. Let us write the left-hand side of the above inequality as

$$\sum_{w_1 \in \mathcal{U}_1} \sum_{w_2' \in \mathcal{U}_2} I_{w_1,w_2'},$$

where

$$I_{w_1,w_2'} = \Big\langle \sum_{w_2 \in \mathcal{U}_2} p_{w_1} p_{w_2}, \sum_{w_1' \in \mathcal{U}_1} p_{w_1'} p_{w_2'} \Big\rangle.$$

Fixing $w_1 \in \mathcal{U}_1$ and $w_2' \in \mathcal{U}_2$, it is enough to show that

$$|I_{w_1,w_2'}| \lesssim R^{c\delta} R^{-(n-1)/2} \min\big(\big|\mathcal{U}_1^{\widetilde{\Pi}_{z_1,z_2}}\big|, \big|\mathcal{U}_2^{\widetilde{\Pi}_{z_1,z_2}}\big|\big).$$

By symmetry we only need to show

$$|I_{w_1,w_2'}| \lesssim R^{c\delta} R^{-(n-1)/2} \big|\mathcal{U}_1^{\widetilde{\Pi}_{z_1,z_2}}\big| \tag{2.8}$$

with $w_1 = (y_1, v_1)$, $w_2' = (y_2', v_2')$, $z_1 = (v_2', \phi(v_2'))$ and $z_2 = (v_1, \phi(v_1))$.

We observe that the Fourier supports of the functions $\sum_{w_2 \in \mathcal{U}_2} p_{w_1} p_{w_2}$, $\sum_{w_1' \in \mathcal{U}_1} p_{w_1'} p_{w_2'}$ are contained in the sets

$$S_2 + z_2 + O\big(R^{-1/2}\big), \qquad S_1 + z_1 + O\big(R^{-1/2}\big),$$

respectively. So it is obvious that

$$\langle p_{w_1} p_{w_2}, p_{w_1'} p_{w_2'} \rangle \neq 0$$

only if $w_1' = (y_1', v_1')$ and $w_2 = (y_2, v_2)$ satisfy

$$z_2 + \big(v_2, \phi(v_2)\big) \in \Pi_{z_1,z_2} + O\big(R^{-1/2}\big),$$
$$z_1 + \big(v_1', \phi\big(v_1'\big)\big) \in \Pi_{z_1,z_2} + O\big(R^{-1/2}\big),$$

and

$$z_2 + \big(v_2, \phi(v_2)\big) = z_1 + \big(v_1', \phi\big(v_1'\big)\big) + O\big(R^{-1/2}\big). \tag{2.9}$$

Therefore, $|I_{w_1,w_2'}|$ is bounded by

$$\sum_{\{w_1' \in \mathcal{U}_1{}^{\widetilde{\Pi}_{z_1,z_2}}\}} \bigg(\sum_{\{w_2 \in \mathcal{U}_2:\ v_1+v_2=v_1'+v_2'+O(R^{-1/2})\}} \big|\langle p_{w_1} p_{w_2}, p_{w_1'} p_{w_2'} \rangle\big| \bigg).$$

Hence it is now sufficient to show that for fixed w_1, w_2', w_1',

$$\sum_{\{w_2 \in \mathcal{U}_2:\ v_2=v_1'+v_2'-v_1+O(R^{-1/2})\}} \big|\langle p_{w_1} p_{w_2}, p_{w_1'} p_{w_2'} \rangle\big| \lesssim R^{c\delta} R^{-(n-1)/2}.$$

Since w_1, w_2', w_1' are given, there are at most $O(1)$ possible v_2 in the inner summation. Note that $\mathcal{U}_2 \subset \mathcal{W}_2(q)$. That is, all the tubes T_{w_2} are passing through $R^{\delta} q$. Hence, there are at most $O(R^{c\delta})$–$w_2 = (y_2, v)$ satisfying $v = v_2$ because $y_2 \in \mathcal{Y}$ are $R^{1/2}$-separated. Using (P4) and the transversality (1.3) between the tubes T_{w_1} and T_{w_2}, it is easy to see that $|\langle p_{w_1} p_{w_2}, p_{w_1'} p_{w_2'} \rangle| \lesssim R^{-(n-1)/2}$. Therefore, we get the desired estimate. □

2.4. *Proof of (2.7)*

It should be noticed that $w_j \in \mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}$ even though it is not explicitly written out. We write the left-hand side of (2.7) as

$$\bigg\| \sum_{w_1 \not\sim b,\ \text{or}\ w_2 \not\sim b} \bigg(\sum_{q \in Q^{\mu_1,\mu_2},\, q \subset 2b} \chi_q p_{w_1} p_{w_2} \bigg) \bigg\|_{L^2(b)}^2$$
$$= \sum_{q \in Q^{\mu_1,\mu_2},\, q \subset 2b} \bigg\| \sum_{w_1 \not\sim b,\ \text{or}\ w_2 \not\sim b} p_{w_1} p_{w_2} \bigg\|_{L^2(q)}^2.$$

Discarding harmless $O(R^{-M})$ terms and using Lemma 2.2, we see that the right-hand side of the above is bounded by

$$CR^{-(n-1)/2+c\delta} \sum_{q\in Q^{\mu_1,\mu_2},\, q\subset 2b} \big|\mathcal{W}_1^{\lambda_1,\mu_1,\mu_2}(q)\big|\big|\mathcal{W}_2^{\lambda_2,\mu_1,\mu_2}(q)\big| \\ \times \Big(\sup_{z_1,z_2} \big|\big[\mathcal{W}_1^{\lambda_1,\mu_1,\mu_2,\not\sim b}(q)\big]^{\widetilde{\Pi}_{z_1,z_2}}\big| + \sup_{z_1,z_2} \big|\big[\mathcal{W}_2^{\lambda_2,\mu_1,\mu_2,\not\sim b}(q)\big]^{\widetilde{\Pi}_{z_1,z_2}}\big| \Big),$$

where $\mathcal{W}_i^{\lambda_i,\mu_1,\mu_2,\not\sim b} = \{w \in \mathcal{W}_i^{\lambda_1,\mu_1,\mu_2}\colon\ w \not\sim b\}$. Since $|\mathcal{W}_j^{\lambda_i,\mu_1,\mu_2}(q)| \lesssim \mu_j$ and

$$\sum_{q\in Q^{\mu_1,\mu_2}} \big|\mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}(q)\big| \leqslant C\lambda_j |\mathcal{W}_j|,^{7}$$

one can easily see that

$$\sum_{q\in Q^{\mu_1,\mu_2},\, q\subset 2b} \big|\mathcal{W}_1^{\lambda_1,\mu_1,\mu_2}(q)\big|\big|\mathcal{W}_2^{\lambda_2,\mu_1,\mu_2}(q)\big| \leqslant C\min\big(\lambda_1\mu_2|\mathcal{W}_1|, \lambda_2\mu_1|\mathcal{W}_2|\big).$$

Therefore, to show (2.7) it is enough to show that if $q \in Q^{\mu_1,\mu_2}$ and $q \subset 2b$, then

$$\lambda_1\mu_2\big|\big[\mathcal{W}_1^{\lambda_1,\mu_1,\mu_2,\not\sim b}(q)\big]^{\widetilde{\Pi}_{z_1,z_2}}\big| \leqslant CR^{\frac{n-1-k}{2}+c\delta}|\mathcal{W}_2|, \\ \lambda_2\mu_1\big|\big[\mathcal{W}_2^{\lambda_2,\mu_1,\mu_2,\not\sim b}(q)\big]^{\widetilde{\Pi}_{z_1,z_2}}\big| \leqslant CR^{\frac{n-1-k}{2}+c\delta}|\mathcal{W}_1|. \tag{2.10}$$

By symmetry it is enough to show one of the estimates. In fact, to prove (2.10) we need only to show the following.

Lemma 2.3. *Let $q_0 \in \mathcal{Q}(R)$ be contained in $Q(0,R)$ and let $\mathcal{Q}$ be a subset of $\mathcal{Q}(R)$. Additionally, let $\mathcal{U}_j$ be subset of $\mathcal{W}_j$ for $j = 1, 2$. Suppose that*

$$\big|\big\{w \in \mathcal{W}_j\colon\ R^\delta q \cap T_w \neq \emptyset\big\}\big| \sim \mu_j \quad \text{for } q \in \mathcal{Q}, \tag{2.11}$$

and for $j = 1, 2$ and some $\lambda_j, \mu_j \geqslant 1$,

$$\big|\big\{q \in \mathcal{Q}\colon\ R^\delta q \cap T_{w_j} \neq \emptyset,\ \operatorname{dist}(q, q_0) \geqslant 2R^{1-\delta}\big\}\big| \gtrsim R^{-c\delta}\lambda_j \tag{2.12}$$

provided $w_j \in \mathcal{U}_j$. For each $q \in \mathcal{Q}$ let us set

$$\mathcal{U}_j(q) = \big\{w_j \in \mathcal{U}_j\colon\ R^\delta q \cap T_{w_j} \neq \emptyset\big\}.$$

Then, there are constants C and c such that

[7] It can be shown by interchanging the order of summation. More precisely, use $\sum_{q\in Q^{\mu_1,\mu_2}} |\mathcal{W}_j^{\lambda_j,\mu_1,\mu_2}(q)| = \sum_{w_j\in W_j^{\lambda_j,\mu_1,\mu_2}} |\{q\colon\ R^\delta T_{w_j} \cap q \neq \emptyset\}|$.

$$\left|\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}\right| \leqslant CR^{\frac{n-1-k}{2}+c\delta}\frac{|\mathcal{W}_2|}{\lambda_1\mu_2},$$
$$\left|\mathcal{U}_2(q_0)^{\widetilde{\Pi}_{z_1,z_2}}\right| \leqslant CR^{\frac{n-1-k}{2}+c\delta}\frac{|\mathcal{W}_1|}{\lambda_2\mu_1}$$

with c independent of δ.

Proof. By symmetry it is enough to show the estimate for $|\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}|$. To begin with, let us set

$$\mathfrak{N}^{\widetilde{\Pi}_{z_1,z_2}} = \left\{\frac{(\nabla\phi(v_1),1)}{|(\nabla\phi(v_1),1)|}:\ \big(v_1,\phi_1(v_1)\big)+z_1 \in \widetilde{\Pi}_{z_1,z_2}+O\big(R^{-1/2}\big)\right\}.$$

We consider the set

$$\Lambda_1 = \left(\bigcup_{w_1\in\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}} R^\delta T_{w_1}\right)\cap\big(Q(0,2R)\setminus Q\big(q_0,R^{1-\delta}\big)\big).$$

From the definition of $\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}$ the associated tubes T_{w_1} have directions which are normal vectors to S_1+z_1 along $\widetilde{\Pi}_{z_1,z_2}$. More precisely, the normal directions of the tubes are contained in the set $\mathfrak{N}^{\widetilde{\Pi}_{z_1,z_2}}$, and all the tubes T_{w_1} meet with $R^\delta q_0$. Hence one can see that Λ_1 is contained in an $R^{1/2+c\delta}$-neighborhood of the conic set

$$\Gamma_1(R) = \big\{t\mathbf{N}:\ \mathbf{N}\in\mathfrak{N}^{\widetilde{\Pi}_{z_1,z_2}},\ R^{1-\delta}\leqslant|t|\leqslant 4R\big\}+q_0.$$

This is actually an isotropic dilation of the set $\Gamma_1+(R^{-\frac12})$ by factor R. Since the surface S has k nonvanishing curvatures, and by (1.4), the normal map $\xi\in\Pi_{z_1,z_2}:\to\mathbf{N}_{z_1}(\xi)\in\mathbb{S}^n$ has rank k. Since Π_{z_1,z_2} has dimension $n-1$, the tubes T_{w_1}, $w_1\in\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}$ can overlap at most $O(R^{c\delta+\frac{n-1-k}{2}})$ over the set Λ. It can be more clearly seen using implicit function theorem. Hence we have

$$\sum_{w\in\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}}\chi_{R^\delta T_w}\leqslant CR^{c\delta+\frac{n-1-k}{2}}. \tag{2.13}$$

By (2.12) we have

$$\lambda_1\left|\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}\right|\leqslant CR^{c\delta}\sum_{w\in\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}}\left|\big\{q\in\mathcal{Q}:\ R^\delta q\cap T_w\neq\emptyset,\ \operatorname{dist}(q,q_0)\geqslant R^{1-\delta}\big\}\right|.$$

Hence it follows that

$$\lambda_1\left|\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}\right|\leqslant CR^{c\delta-\frac{n+1}{2}}\sum_{w\in\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}}\int_\Lambda\chi_{R^\delta T_w}\left(\sum_{q\in\mathcal{Q}}\chi_q\right)dx\,dt.$$

By (2.13) we get

$$\lambda_1 \left|\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}\right| \leqslant CR^{-\frac{n+1}{2}} \int_{\Lambda} \Bigl(\sum_{w\in\mathcal{U}_1(q_0)^{\widetilde{\Pi}_{z_1,z_2}}} \chi_{R^\delta T_w} \Bigr) \Bigl(\sum_{q\in\mathcal{Q}} \chi_q \Bigr)\, dx\, dt$$
$$\leqslant CR^{c\delta+\frac{n-1-k}{2}} \left|\{q\in\mathcal{Q}\colon q\subset\Lambda_1\}\right|.$$

On the other hand it is not difficult to see that the condition (1.5) implies that the tube T_{w_2} meets the opposite tube cone $\Gamma_1(R)$ transversally. Hence we see that T_{w_2} can intersect only $O(R^{c\delta})$ number of $q\subset\Lambda_1$. (See Remark 1.3.) This means that the collections of tubes $\{T_{w_2}\}_{w_2\in\mathcal{W}_2(q)}$ are essentially disjoint along $q\subset\Lambda_1$ (overlapping at most $R^{c\delta}$) so that

$$\sum_{q\in\Lambda_1} \left|\mathcal{W}_2(q)\right| \leqslant CR^{c\delta}|\mathcal{W}_2|.$$

Hence by (2.11) it follows that

$$\mu_2\left|\{q\in\mathcal{Q}\colon q\subset\Lambda_1\}\right| \leqslant C \sum_{q\in\mathcal{Q}\colon q\subset\Lambda_1} \left|\left\{w\in\mathcal{W}_2\colon R^\delta q\cap T_w\neq\emptyset\right\}\right|$$
$$\leqslant CR^{c\delta}|\mathcal{W}_2|.$$

Therefore we get the required inequality comparing the upper and lower bounds for $|\{q\in\mathcal{Q}\colon q\subset\Lambda\}|$. $\square$

3. Applications to linear restriction estimates

In this section we apply Theorem 1.4 to obtain linear restriction estimates for some surfaces with vanishing curvatures. It is also possible to obtain results for more general surfaces but instead we do it with some model surfaces to avoid unnecessary complications.

Let $1\leqslant L\leqslant n-1$ and let $n_1, n_2, \dots, n_L \geqslant 2$ be positive integers satisfying

$$n_1+n_2+\dots+n_L=n.$$

For each $1\leqslant l\leqslant L$, let $\eta^l\in\mathbb{R}^{n_l-1}$ and $\rho^l\in[1,2]$. We set

$$\eta=\left(\eta^1,\dots,\eta^L\right)\in\mathbb{R}^{n-L}, \qquad \rho=\left(\rho^1,\dots,\rho^L\right)\in[1,2]^L.$$

Abusing notation, we sometimes denote $(\rho^1\eta^1, \rho^2\eta^2, \dots, \rho^L\eta^L)$ by

$$\rho\eta=\left(\rho^1\eta^1,\rho^2\eta^2,\dots,\rho^L\eta^L\right)$$

when it is more convenient. Now, the angular variable $\theta=\theta(\xi)$ for $\xi=(\eta,\rho)$ is defined by setting

$$\theta=\left(\theta^1,\theta^2,\dots,\theta^L\right)=\left(\frac{\eta^1}{\rho^1},\frac{\eta^2}{\rho^2},\dots,\frac{\eta^L}{\rho^L}\right)=\frac{\eta}{\rho}\in[1,-1]^{n-L}.$$

Using these notations, we write the variable ξ so that

$$\xi = (\eta, \rho) = (\rho\theta, \rho).$$

Let us set

$$Q = \{(\eta, \rho)\colon\ |\eta| \leqslant 1,\ \rho \in [1, 2]^L\}.$$

Then we consider the smooth conic surface

$$S = \{(\xi, -\phi(\xi))\colon\ \xi \in Q\}$$

with ϕ given by

$$\phi(\xi) = \sum_{l=1}^{L} \pm \frac{|\eta^l|^2}{\rho^l} = \sum_{l=1}^{L} \pm \rho^l |\theta^l|^2. \tag{3.1}$$

Here the sign can be chosen arbitrarily for each summand. Obviously S has L vanishing and $n - L$ nonvanishing curvatures. Instead of dealing with the operator $f \to \widehat{f\,d\sigma}$, we consider as before the equivalent operator $\mathcal{E}_\phi$ given by

$$\mathcal{E}_\phi f(x, t) = \int\limits_Q e^{i(x\cdot\xi - t\phi(\xi))} f(\xi)\, d\xi.$$

We define a mixed norm space by

$$\|f\|_{L^p_\theta L^q_\rho} = \left(\int\limits_{[1,-1]^{n-L}} \left(\int\limits_{[1,2]^L} |f(\rho\theta, \rho)|^q \, d\rho \right)^{p/q} d\theta \right)^{\frac{1}{p}}.$$

Theorem 3.1. *Let ϕ be given by* (3.1). *If $\frac{n-L+2}{q} \leqslant (n-L)(1-\frac{1}{p})$ and $p \geqslant 2$, then, for $q > \frac{2n-2L+8}{n-L+2}$ when $n - L \geqslant 2$, and for $q > 4$ when $n - L = 1$, there is a constant C such that*

$$\|\mathcal{E}_\phi f\|_q \leqslant C\|f\|_{L^p_\theta L^2_\rho}.$$

This is obviously stronger than the usual L^p–L^q estimate when $p > 2$. An interpolation with the trivial estimate $\|\mathcal{E}_\phi f\|_\infty \leqslant C\|f\|_{L^1_\theta L^1_\rho}$ further strengthens the restriction estimate slightly. By adapting Knapp's example one can easily show that the condition $\frac{n-L+2}{q} \leqslant (n-L)(1-\frac{1}{p})$ is necessary. When the surface is the cone, the $L^p_\theta L^2_\rho \to L^q$ estimates were obtained for the non-endpoint case [13].[8] Theorem 3.1 also includes the endpoint case. We prove Theorem 3.1 by obtaining bilinear estimates with suitable separation. The following is what we need.

Proposition 3.2. *Let Q_1, Q_2 be the cubes contained in Q and the extension operators $\mathcal{E}_1$, $\mathcal{E}_2$ be defined by* (2.1). *Suppose that*

$$\left|\frac{\eta_1}{\rho_1}-\frac{\eta_2}{\rho_2}\right|\sim 1 \tag{3.2}$$

(*equivalently* $|\theta_1-\theta_2|\sim 1$) *for each* $(\eta_1,\rho_1)\in Q_1$, $(\eta_2,\rho_2)\in Q_2$. *Then for* $q>\frac{n-L+4}{n-L+2}$,

$$\left\|\prod_{j=1}^{2}\mathcal{E}_j f_j\right\|_q\leqslant C\prod_{j=1}^{2}\|f_j\|_2.$$

Remark 3.3. Adapting the usual counterexample in [10], one can show that the estimate fails for $q<\frac{n-L+4}{n-L+2}$. In fact, for $0<\epsilon\ll 1$ and $i=1,2$, let f_i be the characteristic functions of the sets

$$\left\{(\eta,\rho)\in Q\colon\ \left|\left(\eta^1\right)_1+(-1)^i\right|\leqslant\epsilon^2,\ \left|\rho^1-1\right|\leqslant\epsilon,\ \left|\eta^*\right|\leqslant\epsilon\right\},$$

where $\eta=((\eta^1)_1,\eta^*)\in\mathbb{R}\times\mathbb{R}^{n-L-1}$. Let $x=(y,z)\in\mathbb{R}^{n-L}\times\mathbb{R}^L$, $y=((y^1)_1,y^*)\in\mathbb{R}\times\mathbb{R}^{n-L-1}$ and $z=(z^1,z^*)\in\mathbb{R}\times\mathbb{R}^{L-1}$. Then the condition (3.2) is satisfied on the supports of f_1, f_2, and $|\mathcal{E}f_1\mathcal{E}f_2(x,t)|\sim\epsilon^{2(n-L+2)}$ on the set $\{(x,t)=(y,z,t)\colon |t|\leqslant c\epsilon^{-2},\ |(y^1)_1|\leqslant c\epsilon^{-2},\ |y^*|\leqslant c\epsilon^{-1},\ |z^1-t|\leqslant c\epsilon^{-1},\ |z^*|\leqslant c\}$ for some small $c>0$. The bilinear L^2 estimate implies $\epsilon^{2(n-L+2)}\epsilon^{-(n-L+4)/q}\leqslant C\epsilon^{(n-L+2)}$. Hence letting $\epsilon\to 0$, we see that $q\geqslant\frac{n-L+4}{n-L+2}$.

Assuming Proposition 3.2 for the moment, we prove Theorem 3.1. The argument below is an adaptation of the usual argument which was used to derive linear estimates from bilinear estimates [10]. To obtain the endpoint estimates we also use a simple summation argument involving Lorentz spaces and real interpolation (see, for instance [4]).

Proof of Theorem 3.1. Since f is supported in a fixed compact set, it is enough to show Theorem 3.1 at the endpoint (p,q) that satisfies

$$\frac{n-L+2}{q}=(n-L)\left(1-\frac{1}{p}\right). \tag{3.3}$$

We consider separately the cases $n-L\geqslant 2$ and $n-L=1$ and first prove the case $n-L\geqslant 2$. Since the other case can be handled similarly, we only give a brief remark at the end of proof.

When $n-L\geqslant 2$, due to the known L^2 restriction estimate (1.1) [3,7,11] we may assume that $p>2$ and $q<4$.

To exploit the separation condition we make a decomposition of $\mathcal{E}_\phi f\mathcal{E}_\phi f$. Let $I=[-1,1]^{n-L}$. For each $k\in\mathbb{Z}_+$ we partition I into dyadic cubes Θ_m^k of side 2^{-k}. Here m is the

[8] That is, $\frac{n-L+2}{q}<(n-L)(1-\frac{1}{p})$.

index running over these cubes of side length 2^{-k}. We write $\Theta_m^k \sim \Theta_{m'}^k$ if Θ_m^k and $\Theta_{m'}^k$ have adjacent parents, but are not adjacent. As in [10], we use a Whitney type decomposition of $I \times I$ away from its diagonal D, so that

$$(I \times I \setminus D) = \bigcup_{k=0}^{\infty} \bigcup_{\Theta_m^k \sim \Theta_{m'}^k} \Theta_m^k \times \Theta_{m'}^k.$$

We set

$$f_m^k(\eta, \rho) = \chi_{\Theta_m^k}\left(\frac{\eta}{\rho}\right) f(\eta, \rho) = \chi_{\Theta_m^k}(\theta) f(\rho\theta, \rho).$$

Then it follows that

$$(\mathcal{E} f)(\mathcal{E} f) = \sum_{k \geqslant 0} \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}\big(f_m^k\big)\mathcal{E}\big(f_{m'}^k\big). \tag{3.4}$$

Here we drop the subscript of $\mathcal{E}_\phi$ and do so for the rest of the proof.

Fixing k, we claim that for $q > \frac{2n-2L+8}{n-L+2}$ and $p \geqslant 2$

$$\Big\| \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}\big(f_m^k\big)\mathcal{E}\big(f_{m'}^k\big)\Big\|_{q/2} \leqslant C 2^{2k(\frac{(n-L)}{p} + \frac{(n-L+2)}{q}) - (n-L)} \|f\|^2_{L_\theta^p L_\rho^2}. \tag{3.5}$$

Once one has (3.5), then the desired bound can be shown by using a simple summation argument. In fact, let us fix p_0, q_0 satisfying $q_0 > \frac{2n-2L+8}{n-L+2}$, $p_0 > 2$ and $\frac{n-L+2}{q_0} = (n-L)(1 - \frac{1}{p_0})$. Using (3.5), for $q > \frac{2n-2L+8}{n-L+2}$ we have

$$\Big\| \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}\big(f_m^k\big)\mathcal{E}\big(f_{m'}^k\big)\Big\|_{q/2} \leqslant C 2^{2k(n-L+2)(\frac{1}{q} - \frac{1}{q_0})} \|f\|^2_{L_\theta^{p_0} L_\rho^2}. \tag{3.6}$$

Let K be an integer to be chosen later. Then using (3.4) and splitting the sum $\sum_k$ to $\sum_{-\infty}^{K}$, $\sum_{K+1}^{\infty}$, we see that

$$\begin{aligned} \big|\big\{(x,t)\colon \big|(\mathcal{E} f \mathcal{E} f)(x,t)\big| > \lambda\big\}\big| \leqslant{}& \left|\left\{(x,t)\colon \left|\sum_{-\infty}^{K} \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}\big(f_m^k\big)\mathcal{E}\big(f_{m'}^k\big)\right| > \frac{\lambda}{2}\right\}\right| \\ &+ \left|\left\{(x,t)\colon \left|\sum_{K+1}^{\infty} \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}\big(f_m^k\big)\mathcal{E}\big(f_{m'}^k\big)\right| > \frac{\lambda}{2}\right\}\right|. \end{aligned}$$

Let us choose q_1, q_2 such that $\frac{2n-2L+8}{n-L+2} < q_1 < q_0 < q_2$. Obviously, using triangle inequality and (3.6), by summation of geometric series we get

$$\Big\| \sum_{k=-\infty}^{K} \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}(f_m^k)\mathcal{E}(f_{m'}^k) \Big\|_{q_1/2} \leqslant C 2^{2K(n-L+2)(\frac{1}{q_1}-\frac{1}{q_0})} \|f\|^2_{L_\theta^{p_0} L_\rho^2},$$

$$\Big\| \sum_{k=K+1}^{\infty} \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}(f_m^k)\mathcal{E}(f_{m'}^k) \Big\|_{q_2/2} \leqslant C 2^{2K(n-L+2)(\frac{1}{q_2}-\frac{1}{q_0})} \|f\|^2_{L_\theta^{p_0} L_\rho^2}.$$

Hence by Chebyshev's inequality and the above we have

$$\big|\big\{(x,t):\ \big|(\mathcal{E}f\mathcal{E}f)(x,t)\big| > \lambda\big\}\big| \leqslant C 2^{K(n-L+2)(1-\frac{q_1}{q_0})} \|f\|^{q_1}_{L_\theta^{p_0} L_\rho^2} \lambda^{-q_1/2} + C 2^{K(n-L+2)(1-\frac{q_2}{q_0})} \|f\|^{q_2}_{L_\theta^{p_0} L_\rho^2} \lambda^{-q_2/2}.$$

Then by choosing K which optimizes the right-hand side of the above inequality we get

$$\|\mathcal{E}f\|_{L^{q_0,\infty}} \leqslant C \|f\|_{L_\theta^{p_0} L_\rho^2}.$$

Finally, this estimate is valid for p_0, q_0 satisfying $q_0 > \frac{2n-2L+8}{n-L+2}$, $p_0 > 2$ and (3.3). Real interpolation among these estimates gives the desired because $p_0 \leqslant q_0$. Hence now it remains to show (3.5).

Observe that for a fixed k the Fourier transforms of $\mathcal{E}(f_m^k)\mathcal{E}(f_{m'}^k)$, $\Theta_m^k \sim \Theta_{m'}^k$ are supported in boundedly overlapping parallelepipeds. Hence, for $2 \leqslant q \leqslant 4$ we have

$$\Big\| \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}(f_m^k)\mathcal{E}(f_{m'}^k) \Big\|_{q/2} \leqslant C \Big(\sum_{\Theta_m^k \sim \Theta_{m'}^k} \big\|\mathcal{E}(f_m^k)\mathcal{E}(f_{m'}^k)\big\|_{q/2}^{q/2} \Big)^{2/q}. \tag{3.7}$$

This follows from interpolation between L^2 and trivial L^1 estimates (see Lemma 6.1 in [10]). By this, we are reduced to showing that if $\Theta_m^k \sim \Theta_{m'}^k$, then

$$\big\|\mathcal{E}(f_m^k)\mathcal{E}(f_{m'}^k)\big\|_{q/2} \leqslant C 2^{2k(\frac{(n-L)}{p}+\frac{(n-L+2)}{q})-(n-L)} \|f_m^k\|_{L_\theta^p L_\rho^2} \|f_{m'}^k\|_{L_\theta^p L_\rho^2}. \tag{3.8}$$

In fact, putting this in the right-hand side of the above, we have

$$\Big\| \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}(f_m^k)\mathcal{E}(f_{m'}^k) \Big\|_{q/2} \leqslant C 2^{2k(\frac{(n-L)}{p}+\frac{(n-L+2)}{q})-(n-L)} \times \Big(\sum_{\Theta_m^k \sim \Theta_{m'}^k} \|f_m^k\|^{q/2}_{L_\theta^p L_\rho^2} \|f_{m'}^k\|^{q/2}_{L_\theta^p L_\rho^2} \Big)^{2/q}.$$

Therefore the desired inequality (3.5) follows if one can show that

$$\left(\sum_{\Theta_m^k\sim\Theta_{m'}^k}\|f_m^k\|_{L_\theta^pL_\rho^2}^{q/2}\|f_{m'}^k\|_{L_\theta^pL_\rho^2}^{q/2}\right)^{2/q}\leqslant C\|f\|_{L_\theta^pL_\rho^2}^{q/2}\|f\|_{L_\theta^pL_\rho^2}^{q/2}.$$

By Cauchy–Schwarz's inequality, the left-hand side is bounded by $(\sum_m\|f_m^k\|_{L_\theta^pL_\rho^2}^q)^{2/q}$. It is again bounded by $\|f\|_{L_\theta^pL_\rho^2}^2$ since $q\geqslant p$ and the angular projections of the supports of f_m^k are disjoint.

Now we proceed to show (3.8). Let $\Theta_1,\Theta_2\subset[-1,1]^{n-L}$ be cubes of $\operatorname{diam}(\Theta_1),\operatorname{diam}(\Theta_2)\leqslant 2^{-k}$, and $\operatorname{dist}(\Theta_1,\Theta_2)\geqslant 2^{-k}$. For $j=1,2$, let us set

$$P\big(\Theta_j,2^{-k}\big)=\big\{(\eta,\rho)\in Q\colon\ \eta=\rho\theta,\ \theta\in\Theta_j\big\}.$$

For (3.8) we need show that

$$\|\mathcal{E}f\mathcal{E}g\|_{q/2}\leqslant C2^{2k(\frac{(n-L)}{p}+\frac{(n-L+2)}{q})-(n-L)}\|f\|_{L_\theta^pL_\rho^2}\|g\|_{L_\theta^pL_\rho^2}\tag{3.9}$$

whenever f, g are supported in $P(\Theta_1,2^{-k})$, $P(\Theta_2,2^{-k})$, respectively.

Let $\theta_0=(\theta_0^1,\theta_0^2,\dots,\theta_0^L)\in\mathbb{R}^{n-L}$ be the center of the smallest cube containing both of Θ_1,Θ_2. We write the phase part of the extension operator $\mathcal{E}f$ as

$$x\cdot\xi-t\phi(\xi)=\tilde{x}\cdot\eta+\bar{x}\cdot\rho-t\left(\sum_{l=1}^{L}\pm\frac{|\eta^l|^2}{\rho^l}\right)$$

where $x=(\tilde{x},\bar{x})\in\mathbb{R}^{n-L}\times\mathbb{R}^L$. Let us set $\tilde{x}=(\tilde{x}^1,\tilde{x}^2,\dots,\tilde{x}^L)$, $\tilde{x}^l\in\mathbb{R}^{n_l-1}$ and $\bar{x}=(\bar{x}^1,\bar{x}^2,\dots,\bar{x}^L)$, $\bar{x}^l\in\mathbb{R}$. The phase part is transformed to

$$(\tilde{x}\mp 2t\theta_0)\cdot\eta+\sum_{l=1}^{L}\big(\tilde{x}^l\cdot\theta_0^l+\bar{x}^l\mp t\big|\theta_0^l\big|^2\big)\rho^l-t\left(\sum_{l=1}^{L}\pm\frac{|\eta^l|^2}{\rho^l}\right)$$

under the change of variables

$$(\eta,\rho)\to(\eta+\rho\theta_0,\rho)=\big(\eta^1+\theta_0^1\rho^1,\eta^2+\theta_0^2\rho^2,\dots,\eta^L+\theta_0^L\rho^L,\rho\big).$$

Note that this sends the parallelepipeds $P(\Theta_1,2^{-k})$, $P(\Theta_2,2^{-k})$ to $P(\Theta_1^*,2^{-k})$, $P(\Theta_2^*,2^{-k})$, respectively. Here Θ_1^*, Θ_2^* are cubes contained in $[-2^{-k+1},2^{-k+1}]^{n-L}$ with $\operatorname{dist}(\Theta_1^*,\Theta_2^*)\geqslant 2^{-k}$. Performing the change of variables $(\eta,\rho)\to(\eta+\rho\theta_0,\rho)$ for both of the integrals $\mathcal{E}(f)$, $\mathcal{E}(g)$, we see

$$(\mathcal{E}f\mathcal{E}g)(x,t)=(\mathcal{E}\tilde{f}\mathcal{E}\tilde{g})(Tx,t),$$

where $\tilde f$, $\tilde g$ are functions satisfying that $\|\tilde f\|_{L^p_\theta L^2_\rho} = \|f\|_{L^p_\theta L^2_\rho}$, $\|\tilde g\|_{L^p_\theta L^2_\rho} = \|g\|_{L^p_\theta L^2_\rho}$, and

$$T(x) = \big(\tilde x \mp 2t\theta, \tilde x^1 \cdot \theta_0^1 + \bar x^1 \mp t\big|\theta_0^1\big|^2, \dots, \tilde x^L \cdot \theta_0^L + \bar x^L \mp t\big|\theta_0^L\big|^2\big).$$

Since $|\det T| = 1$, the matters are reduced to showing that

$$\|\mathcal{E} f \mathcal{E} g\|_{q/2} \leqslant C 2^{2k(\frac{(n-L)}{p} + \frac{(n-L+2)}{q}) - (n-L)} \|f\|_{L^p_\theta L^2_\rho} \|g\|_{L^p_\theta L^2_\rho}$$

whenever f, g are supported in $P(\Theta_1^*, 2^{-k})$, $P(\Theta_2^*, 2^{-k})$. Now we make an additional change of variables

$$(\eta, \rho) \to \big(2^{-k}\eta, \rho\big)$$

for both of the integrals $\mathcal{E} f$, $\mathcal{E} g$ to see we see

$$(\mathcal{E} f \mathcal{E} g)(x, t) = 2^{-2k(n-L)} (\mathcal{E} f_k \mathcal{E} g_k)\big(2^{-k}\tilde x, \bar x, 2^{-2k} t\big)$$

where f_k, g_k are functions satisfying that

$$\|f_k\|_{L^p_\theta L^2_\rho} = 2^{k(n-L)/p} \|f\|_{L^p_\theta L^2_\rho}, \qquad \|g_k\|_{L^p_\theta L^2_\rho} = 2^{k(n-L)/p} \|g\|_{L^p_\theta L^2_\rho}$$

and the supports of f_k, g_k are contained in $P(\Theta_1^o, \frac12)$, $P(\Theta_2^o, \frac12)$, satisfying $\Theta_1^o, \Theta_2^o \subset [-1, 1]^{n-L}$ and $\operatorname{dist}(\Theta_1^o, \Theta_2^o) \geqslant 1$. Then, after rescaling, it is sufficient to show that

$$\|\mathcal{E} f_k \mathcal{E} g_k\|_{q/2} \leqslant C \|f_k\|_{L^p_\theta L^2_\rho} \|g_k\|_{L^p_\theta L^2_\rho}$$

provide that f_k, g_k are supported in $P(\Theta_1^o, \frac12)$, $P(\Theta_2^o, \frac12)$, respectively. This follows from Proposition 3.2. Hence we get (3.5). This completes the proof of Theorem 3.1 for the case $n - L \geqslant 2$.

Now we turn to the remaining case $n - L = 1$. This condition implies that $n = 2$ and $L = 1$. We only need to show (3.5) in a neighborhood of the segment determined by $3/q + 1/p = 1$, $q > 4$, because once this is obtained the rest of the argument works without modification. Since $4 < q$, we need to replace (3.7) with

$$\Big\| \sum_{\Theta_m^k \sim \Theta_{m'}^k} \mathcal{E}\big(f_m^k\big)\mathcal{E}\big(f_{m'}^k\big) \Big\|_{q/2} \leqslant C \Big(\sum_{\Theta_m^k \sim \Theta_{m'}^k} \big\|\mathcal{E}\big(f_m^k\big)\mathcal{E}\big(f_{m'}^k\big)\big\|_{(q/2)}^{(q/2)'} \Big)^{(2/q)'}$$

which is valid for $q \geqslant 4$. Then, using (3.9), one can get the desired estimate (3.5) as long as $2(q/2)' \geqslant p$. There is a small neighborhood of the segment determined by $3/q + 1/p = 1$, $q > 4$, where this condition is satisfied. $\square$

Proof of Proposition 3.2. To prove Proposition 3.2, we use Theorem 1.4. It is enough to verify the conditions (1.3), (1.4) and (1.5) when the surface is given by $(\xi, \phi(\xi))$ and (3.2) is satisfied. We only show it for the case

$$\phi(\xi) = \sum_{l=1}^{L} \frac{|\eta^l|^2}{\rho^l}.$$

It is not difficult to see that the same is valid for the other cases.

The geometry of Π_{z_1,z_2} can be complicated and the conditions (1.3), (1.4) and (1.5) are not so easy to check directly in practical applications. So we simplify the picture by projecting Π_{z_1,z_2} to $\mathbb{R}^n$. It generally makes some information be lost but in our example we can still get the sharp results. By finite decomposition we may assume that Q_1 and Q_2 are as small as we wish. More precisely, let $(\rho_1\theta_1, \rho_1), (\rho_2\theta_2, \rho_2) \in Q$. Then, we may assume that the surfaces are given as graphes of ϕ over the sets

$$Q_1 = \{(\rho\theta, \rho) \in Q \colon |\theta - \theta_1| < \epsilon_0,\ |\rho - \rho_1| < \epsilon_0\},$$
$$Q_2 = \{(\rho\theta, \rho) \in Q \colon |\theta - \theta_2| < \epsilon_0,\ |\rho - \rho_2| < \epsilon_0\},$$

respectively, for some small $\epsilon_0 > 0$. Hence by continuity, to verify the conditions (1.3), (1.4) and (1.5) it is enough to show them at each point $(\rho_1\theta_1, \rho_1), (\rho_2\theta_2, \rho_2) \subset Q$ if ϵ_0 is sufficiently small.

Let us denote by π_{z_1,z_2} the projection of Π_{z_1,z_2} to spatial space. That is, $\pi_{z_1,z_2} = \{v \colon (v, \tau) \in \Pi_{z_1,z_2}\}$. Let us write $z_i = (v_i, \tau_i)$ for $i = 1, 2$. Since $S_{z_i} = \{(u, \phi(u)) + z_i \colon u \in Q_i\}$, $i = 1, 2$, one can easily see

$$\pi_{z_1,z_2} = \{v \in (Q_1 + v_1) \cap (Q_2 + v_2) \colon \phi(v - v_1) - \phi(v - v_2) = \tau_2 - \tau_1\}.$$

Hence π_{z_1,z_2} is an $(n-1)$-dimensional immersed surface as long as $\nabla\phi(v-v_1) - \nabla\phi(v-v_2) \neq 0$ for all $v \in \pi_{z_1,z_2}$. Obviously this condition is equivalent to (1.3). Now note that

$$\nabla\phi(\rho\theta, \rho) = (2\theta^1, 2\theta^2, \ldots, 2\theta^L, -|\theta^1|^2, \ldots, -|\theta^L|^2).$$

So we have for $\xi_1 = (\rho_1\theta_1, \rho_1)$, $\xi_2 = (\rho_2\theta_2, \rho_2)$,

$$\nabla\phi(\xi_1) - \nabla\phi(\xi_2) = (2\theta_1 - 2\theta_2, |\theta_2^1|^2 - |\theta_1^1|^2, \ldots, |\theta_2^L|^2 - |\theta_1^L|^2).$$

Hence, the condition (1.3) is satisfied by (3.2).

Note that at the point $\xi = (\rho\theta, \rho)$ the projected null directions[9] are given the span of the vectors

$$\mathcal{N}^1(\theta) = (\theta^1, 0, \ldots, 0, 1, 0, \ldots, 0),$$
$$\mathcal{N}^2(\theta) = (0, \theta^2, 0, \ldots, 0, 0, 1, 0, \ldots, 0),$$
$$\vdots$$
$$\mathcal{N}^L(\theta) = (0, \ldots, 0, \theta^L, 0, \ldots, 1).$$

[9] That is, $(\mathcal{N}^j \cdot \nabla)^2\phi = 0$.

The condition (1.4) is satisfied as long as at least one of the null directions from each surface is transversal to the tangent space of $\varPi_{z_1,z_2}$. Since the surface π_{z_1,z_2} has dimension $n-1$, (1.4) is satisfied if one of the projected null directions $\mathcal{N}^1(\theta_1), \mathcal{N}^2(\theta_1), \dots, \mathcal{N}^L(\theta_1)$ is transversal to the tangent space of π_{z_1,z_2}. On the other hand the normal vector to π_{z_1,z_2} is $\nabla\phi(\rho_1\theta_1,\rho_1) - \nabla\phi(\rho_2\theta_2,\rho_2) + O(\epsilon_0)$. So it is enough to show that there are some i, k such that

$$\bigl|\bigl\langle\nabla\phi(\rho_1\theta_1,\rho_1) - \nabla\phi(\rho_2\theta_2,\rho_2), \mathcal{N}^i(\theta_1)\bigr\rangle\bigr| \sim 1,$$
$$\bigl|\bigl\langle\nabla\phi(\rho_1\theta_1,\rho_1) - \nabla\phi(\rho_2\theta_2,\rho_2), \mathcal{N}^k(\theta_2)\bigr\rangle\bigr| \sim 1.$$

By direct computation one can easily see that the right-hand side of the first expression is equal to $|\theta_1^i - \theta_2^i|^2$ and that of the second is $|\theta_1^k - \theta_2^k|^2$. Due to the condition (3.2), there must be i, k (in fact, $i = k$) satisfying the above. Hence (1.4) follows.

Now it remains to show (1.5). By Remark 1.3 we need to show that for $j = 1, 2$, the normal vectors $\mathbf{N}_j$ of surfaces S_{z_j} are transversal to the opposite tube cones $\varGamma_{3-j}$. Instead of considering the sets $\varGamma_1$, $\varGamma_2$, we consider larger sets which are given by

$$\widetilde{\varGamma}_j = \bigl\{t\bigl(\nabla\phi(v), 1\bigr)\colon v \in Q_j,\ 2^{-1} \leqslant |t| \leqslant 2\bigr\}.$$

Obviously $\varGamma_j \subset \widetilde{\varGamma}_j$. So, if the opposite normal $\mathbf{N}_j$ is transversal to $\widetilde{\varGamma}_{3-j}$, it is also transversal to $\varGamma_{3-j}$. By the homogeneity of ϕ

$$\widetilde{\varGamma}_j = \bigl\{t\varPhi(\theta)\colon \theta_j \in \varTheta_j,\ 2^{-1} \leqslant |t| \leqslant 2\bigr\}$$

where $\varTheta_j = \{\frac{\eta}{\rho}\colon (\eta,\rho) \in Q_j\}$ and

$$\varPhi(\theta) = t\bigl(2\theta^1, 2\theta^2, \dots, 2\theta^L, -\bigl|\theta^1\bigr|^2, \dots, -\bigl|\theta^L\bigr|^2, 1\bigr).$$

We only consider the case $j = 2$ because the other case $j = 1$ is obvious from symmetry.

From the above the tangent space of $\widetilde{\varGamma}_1$ at each point is spanned by the partial derivatives $(\partial_{(\theta^l)_k}\varPhi(\theta_1), 0)$, and $(\varPhi(\theta_1), 1)$. Here $(\theta^l)_k$ denotes the k-th component of θ^l. The direction of opposite tube is also given by $(\varPhi(\theta_2), 1)$. So the transversality between $\mathbf{N}_1$ and $\widetilde{\varGamma}_2$ follows if one can show that these $n - L + 1$ vectors $(\partial_{(\theta^L)_k}\varPhi(\theta_1), 0)$, and $\varPhi(\theta_1) - \varPhi(\theta_2)$ are linearly independent. Equivalently, we need to show that the $n \times (n - L + 1)$ matrix

$$M = \bigl[\, \partial_\theta\varPhi(\theta_1), \varPhi(\theta_1) - \varPhi(\theta_2) \,\bigr]$$

has a minor of size $(n - L + 1) \times (n - L + 1)$ with nonvanishing determinant. Here we write $\varPhi(\theta_1)$, $\varPhi(\theta_2)$ as column vectors and $\partial_\theta\varPhi(\theta_1)$ is an $n \times (n - L)$ matrix. If one removes the last $L - 1$ rows from M except $(n - L + l)$-th, then we have the matrix

$$\widetilde{M}_l = \begin{bmatrix} 2I_{n-L} & u \\ v & \alpha \end{bmatrix}$$

where I_{n-L} is the identity matrix of $(n-L)\times(n-L)$, $u = 2\theta_1 - 2\theta_2$, $\alpha = |\theta_2^l|^2 - |\theta_1^l|^2$ and $v = (0,\dots,0,-2\theta_1^l,0,\dots,0)$. By a simple computation one see

$$\det \widetilde{M}_l = 2^{n-L-1}\big(2\alpha - \langle u, v\rangle\big) = 2^{n-L}\big|\theta_1^l - \theta_2^l\big|^2.$$

Therefore there is an $(n-L+1)\times(n-L+1)$ minor $\widetilde{M}_l$ with nonzero determinant because $\det \widetilde{M}_1,\dots,\det \widetilde{M}_L$ cannot be zero simultaneously by the condition (3.2). This completes the proof of Proposition 3.2. □

Since the estimates are stable under a small smooth perturbation of the surface, it is possible to obtain the same results for other surfaces which are of similar type. For example, let us consider

$$\psi(\xi) = \sum_{l=1}^{L}\big|\xi^l\big|$$

where $\xi = (\xi^1,\dots,\xi^L) \in \mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_L}$. Then let $\mathcal{E}_\psi$ be the operator given by

$$\mathcal{E}_\psi f(x,t) = \int\limits_{\{|\xi^l|\sim 1,\,1\leqslant l\leqslant L\}} e^{i(x\cdot\xi - t\phi(\xi))} f(\xi)\,d\xi.$$

We now split variable ξ^l and write $\xi^l = (\eta^l,\rho^l) \in \mathbb{R}^{n_l-1}\times\mathbb{R}$. By a finite decomposition of Q and rotation in each ξ^l one may assume

$$Q = \big\{\xi = \big(\xi^1,\dots,\xi^L\big)\colon \big|\eta^l\big| \leqslant \epsilon_0,\ \rho^l \sim 1,\ 1\leqslant l\leqslant L\big\}$$

with small $\epsilon_0 > 0$. Then by the linear change of variables

$$(x,t) \to \big(\tilde{x}^1 - t\bar{x}^1, \tilde{x}^2 - t\bar{x}^2, \dots, \tilde{x}^L - t\bar{x}^L, t\big)$$

ψ is replaced by $\tilde{\psi}(\xi) = \sum_{l=1}^{L} |(\eta^l,\rho^l)| - \rho^l$. Then by Taylor's expansion

$$\tilde{\psi}(\xi) = \frac{1}{2}\sum_{l=1}^{L}\bigg(\frac{|\eta^l|^2}{\rho^l} + O\big(\big|\eta^l\big|^4\big)\bigg) = \frac{1}{2}\sum_{l=1}^{L}\rho_l\big(\big|\theta^l\big|^2 + O\big(\big|\theta^l\big|^4\big)\big).$$

This is a small smooth perturbation of ψ. Hence the bilinear estimate can be similarly formulated for $\mathcal{E}_\psi$ because all the required (1.3), (1.4) and (1.5) are stable under smooth perturbation. By this stability the *bilinear* $\to$ *linear* argument also works. Let r^l, θ^l be the spherical coordinates for ξ^l such that $\xi^l = r^l\theta^l$ and define a mixed norm

$$\|f\|_{L^p_\theta L^q_r} = \bigg(\int\limits_{\mathbb{S}^{n_1-1}\times\cdots\times\mathbb{S}^{n_L-1}}\bigg(\int\limits_{[1,2]^L}\big|f(r\theta)\big|^q\,dr\bigg)^{p/q} d\theta\bigg)^{\frac{1}{p}}$$

where $\theta = (\theta^1,\dots,\theta^L)$ and $r = (r^1,\dots,r^L)$. Then we get the following.

Corollary 3.4. *Let ψ be the function given as in the above. If $\frac{n-L+2}{q} \leqslant (n-L)(1-\frac{1}{p})$ and $p \geqslant 2$, then for $q > \frac{2n-2L+8}{n-L+2}$ when $n-L \geqslant 2$, and for $q > 4$ when $n-L=1$, there is a constant C such that*

$$\|\mathcal{E}_\psi f\|_q \leqslant C\|f\|_{L^p_\theta L^2_r}.$$

Remark 3.5. Proposition 3.2 can be further generalized. Let $A^1, \dots, A^L$ be nonsingular symmetric matrices of $(n_l-1)\times(n_l-1)$. Then consider

$$\phi(\xi) = \sum_{l=1}^{L} \rho^l \langle A^l \theta^l, \theta^l \rangle,$$

where $\theta^l = \eta^l/\rho^l$. Here we use the same notations which are used in Proposition 3.2. Then the same bilinear estimate holds under the conditions

$$\sum_{l}^{L} \left|\left\langle A^l\left(\theta_1^l - \theta_2^l\right), \left(\theta_1^l - \theta_2^l\right)\right\rangle\right| \sim 1$$

for $(\rho_1\theta_1, \rho_1) \in Q_1$, $(\rho_2\theta_2, \rho_2) \in Q_2$. This is a generalization of the bilinear restriction estimate for a conic surface which was studied in [5]. However, when A^l has eigenvalues of different signs it is still in question whether the sharp linear estimates can be obtained from these bilinear estimates.

References

[1] J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis, Geom. Funct. Anal. 1 (1991) 147–187.

[2] J. Bourgain, Estimates for cone multipliers, in: Geometric Aspects of Functional Analysis, Israel, 1992–1994, in: Oper. Theory Adv. Appl., vol. 77, Birkhäuser, Basel, 1995, pp. 41–60.

[3] A. Greenleaf, Principal curvature and harmonic analysis, Indiana Univ. Math. J. 30 (1981) 519–537.

[4] S. Lee, Endpoint estimates for the circular maximal function, Proc. Amer. Math. Soc. 131 (5) (2003) 1433–1442.

[5] S. Lee, Bilinear restriction estimates for surfaces with curvatures of different signs, Trans. Amer. Math. Soc. 358 (8) (2006) 3511–3533.

[6] G. Mockenhaupt, A note on the cone multiplier, Proc. Amer. Math. Soc. 117 (1) (1993) 145–152.

[7] R. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44 (3) (1977) 705–714.

[8] T. Tao, A sharp bilinear restriction estimate for paraboloids, Geom. Funct. Anal. 13 (2003) 1359–1384.

[9] T. Tao, A. Vargas, A bilinear approach to cone multipliers. I. Restriction estimates, Geom. Funct. Anal. 10 (2000) 185–215.

[10] T. Tao, A. Vargas, L. Vega, A bilinear approach to the restriction and Kakeya conjecture, J. Amer. Math. Soc. 11 (1998) 967–1000.

[11] P. Tomas, A restriction theorem for the Fourier transform, Bull. Amer. Math. Soc. 81 (1975) 477–478.

[12] A. Vargas, Restriction theorems for a surface with negative curvature, Math. Z. 249 (1) (2005) 97–111.

[13] T. Wolff, A sharp cone restriction estimate, Ann. of Math. 153 (2001) 661–698.

Further reading

[14] B. Barcelo, On the restriction of the Fourier transform to a conical surface, Trans. Amer. Math. Soc. 292 (1985) 321–333.
[15] T. Tao, A. Vargas, A bilinear approach to cone multipliers. II. Application, Geom. Funct. Anal. 10 (2000) 216–258.